

A New Control Framework for Flapping-Wing Vehicles based on 3D Pendulum Dynamics

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Abstract

In this paper, a new control framework for an insect-scale flapping-wing vehicle is presented that exploits passive aerodynamic effects to stabilize the attitude dynamics. Many flapping-wing robotic flyers and flying insects share a common morphological feature in that the center of mass (CoM) is below the center of pressure (CoP), which makes the hovering configuration intrinsically unstable with open-loop control. Motivated by the fact that the CoM should be ahead of the CoP to ensure the longitudinal stability of the flight dynamics, a new coordinate system is proposed by placing a virtual control point (VCP) above the CoP. The dynamics in the new coordinates are derived using a near-identity diffeomorphism which admits a partial feedback linearization with stable zero dynamics. The behavior of the zero dynamics resembles the dynamics of a 3D pendulum with an aerodynamic damper. An adaptive controller is proposed to make the upright orientation almost globally asymptotically stable over a bounded uncertainty of the aerodynamic drag coefficient. The controller is evaluated in simulation with a Harvard RoboBee following a virtual control point reference trajectory.

Key words: application of nonlinear analysis and design; flapping-wing vehicle; partial feedback linearization; minimum phase; adaptive control; tracking; 3D Pendulum.

1 Introduction

Over recent decades, research in the field of small-scaled flapping-wing micro aerial vehicles (FWMAV) has rapidly expanded, inspired by the ability of flying insects to perform aggressive maneuvers with seemingly effortless ease. Analysis of the morphological parameters of various flying insects has shown that the wing base attached to the body is often higher than the insect's center of mass (CoM) [8]. This unique feature is reflected in many insect-scale FWMAV, including the Harvard RoboBee (81mg and 170Hz flapping frequency) [21], the CMU piezoelectric-driven FWMAV (160mg and resonance at 35Hz flapping frequency) [12], and the motor driven robotic HummingBird prototype (12g and 34Hz flapping frequency) described in [32].

Past research suggests that the vertical displacement between the CoM and the wing base introduces a body pitch oscillation which renders hovering intrinsically unstable, resulting from the aerodynamic drag acting on each wing [5,25,30]. The open-loop instability of hovering has been extensively studied in model-based simulations, which have demonstrated this effect using techniques including the application of differential geometric higher-order averaging [29], the use of Floquet theory to analyze hovering stability in periodic orbits [27], and simplification via disregarding the wing inertial effects [5,25,30]. See [28] for surveys of control methods.

Prior work on hovering control for FWMAV has leveraged generalized averaging theory [29] and geometric methods similar to those used in standard quadrotor control [21,33]. These controllers operate by suppressing the unstable part of the dynamics to locally stabilize while hovering. (An extensive review of the control architectures of FWMAV can be found in [28].) In particular, previous controllers for the Harvard RoboBee have treated the instability resulting from the separation between the CoM and the center of pressure (CoP) (on average at the wing base of the vehicle, as in Fig. 1) as an

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attitude disturbance, rejecting it with other force and torque disturbances experienced in flight. Techniques used to stabilize vehicle pitch in this manner include the addition of a rotational damper to achieve passive attitude stability [30], the application of a simple torque proportional to the angular velocity to stabilize the upright orientation [11], and the design of an adaptive sliding mode controller to reject unmodeled disturbances (including those due to CoM and CoP separation) [6].

Instead of rejecting the aerodynamic drag effects, we present a new approach designed to take advantage of the drag in order to stabilize the upright orientation of FW-MAV. This idea is motivated by the principle that the CoM should lie above the CoP to obtain static stability in flight [1]. A controller is designed to stabilize a virtual control point (VCP) along the body z axis above the vehicle, as shown in Fig. 1. The system is then analogous to a 3D pendulum, where the VCP serves as the pivot point, and the stroke-averaged aerodynamic damping at the CoP dampens attitude oscillations (since the pivot point lies above the CoP). This is understood by considering the near-identity diffeomorphism introduced in [24], which controls nonholonomic $SE(2)$ vehicles. Both the proposed near-identity diffeomorphism in this paper and that in [24] allow partial feedback linearization with stable zero dynamics. However, the stability proof in [24] cannot be directly transferred, as its asymptotic stability comes from a nonholonomic constraint and there are no such nonholonomic constraints for the dynamics considered herein. The proposed controller guarantees almost global asymptotic stability (AGAS) to the desired invariant set (the hovering configuration), similar to control of a 3D pendulum with a fixed base pivot [4]. Additionally, the VCP-based controller is advantageous in its reduced requirement for yaw torque during trajectory tracking, which is commonly a weak controlled torque for FWMAV [2,9,31,13]. An adaptive controller is also proposed to show robustness over bounded uncertainty in the aerodynamic drag coefficient using the projection-based adaptation in [18]. This proposed adaptive controller also preserves the AGAS property. Finally, an application is demonstrated in the design of a tracking controller, following a reference trajectory with the VCP in simulation.

2 Modeling

We consider a standard rigid body dynamics model (six degrees of freedom) with a stroke-averaged aerodynamic force, controlled via the thrust and torque acting on the rigid body. This model, and underlying assumptions, is appropriate for insect-scale FWMAVs due to their relatively small wing mass. We use the Harvard RoboBee as an example system for this work. The vehicle flies via amplitude-modulated harmonic voltage signals which each drive a piezoelectric bimorph actuator with a resonant frequency of approximately 170Hz, and has a wing

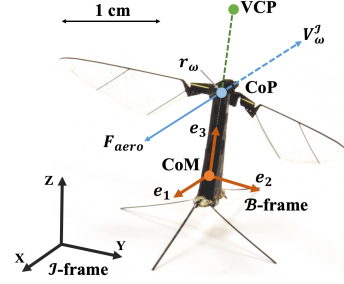


Fig. 1. Harvard RoboBee coordinates.

mass less than 1% of its the total body weight [10]. Previous control approaches for this system [21,7] have found desired thrust and torques, then converted these to voltage signals using a predefined mapping [21].

Let the \mathcal{I} -frame and \mathcal{B} -frame represent the inertial and body-fixed frames, respectively, as shown in Fig. 1. The origin of the \mathcal{B} -frame is located at the center of mass (CoM) position and its orientation aligns with the principal axis of the vehicle, such that its z -direction coincides with the direction of thrust. The configurations of a rigid body can be represented by the special Euclidean group, $SE(3)$, which is a semi-direct product of \mathbb{R}^3 and the special orthogonal group $SO(3)$. In this paper, (r, R) denotes a homogeneous coordinates [22] of $SE(3) \cong \mathbb{R}^3 \rtimes SO(3)$, in which r represents the origin of \mathcal{B} -frame in the \mathcal{I} -frame, and R represents the orientation of \mathcal{B} -frame relative to the \mathcal{I} -frame.

Let v be the velocity of the CoM in the \mathcal{I} -frame, $\omega \in \mathbb{R}^3$ be the instantaneous body angular velocity, m be the total mass, g be the gravity, and I be the moment of inertia. By using the conventional hat operation, $\hat{\cdot} : \mathbb{R}^3 \rightarrow so(3)$, where $so(3) \subset \mathbb{R}^{3 \times 3}$ is the space of skew-symmetric matrices, the full rigid body dynamics are

$$\dot{r} = v \quad (1)$$

$$\dot{R} = R\hat{\omega} \quad (2)$$

$$\dot{v} = -ge_3 + TR e_3 + F_{aero}/m \quad (3)$$

$$I\dot{\omega} = -\omega \times I\omega + \tau + \tau_{aero}, \quad (4)$$

where \times is a cross product, $e_3 = (0, 0, 1)^T$ is an elementary unit vector in \mathbb{R}^3 , T is a thrust normalized by the mass, and $\tau = (\tau_1, \tau_2, \tau_3)$ is an external torque. The dynamics in (3-4) contain the aerodynamic drag force, F_{aero} , and the induced torque, τ_{aero} . The moment of inertia, $I \in \mathbb{R}^{3 \times 3}$, is a diagonal matrix with (I_{xx}, I_{yy}, I_{zz}) as its diagonal components and a skew symmetric matrix, $\hat{\omega}$, is given by a relation, $\hat{\omega}y = \omega \times y$, and $\bar{0} \in \mathbb{R}^3$ represents the zero vector.

2.1 Stroke-Averaged Aerodynamic Damping Model

Wind tunnel experiments [30] have shown that the stroke-averaged aerodynamic drag is approximately a

linear function of the velocity of the center of pressure point of the wing. Let $b_w > 0$ be the stroke-averaged aerodynamic drag coefficient. By assuming the symmetric configuration of both wings, the stroke-averaged aerodynamic drag model can be formulated as

$$F_{aero} = -b_w(v + R(\omega \times r_w)) \quad (5)$$

$$\tau_{aero} = r_w \times R^\top F_{aero}, \quad (6)$$

where $r_w := le_3$ is the centroid of the two wing CoPs in \mathcal{B} -frame, and l is the distance to the CoM. The centroid of the CoP is shown as a blue circle in Fig. 1, and the CoM is shown as a red circle in Fig. 1. The velocity of the CoP, $V_w^\mathcal{I}$, and F_{aero} are also shown in the figure. The exact scale can be found in [11].

3 Preliminary Notions of Set-Stability

Let $D := TSE(3)$ be the domain of (1-4), with $TSE(3)$ the tangent bundle of $SE(3)$. Suppose that $R_1, R_2 \in SO(3)$, then

$$d_R(R_1, R_2) := \|I_{3 \times 3} - R_1^\top R_2\|_F, \quad (7)$$

where $\|\cdot\|_F$ is a Frobenious norm in $\mathbb{R}^{3 \times 3}$, is a metric on $SO(3)$. See [15] for the topological equivalence between different metrics of $SO(3)$. Suppose that $x = (r_x, R_x, v_x, \omega_x) \in D$ and $y = (r_y, R_y, v_y, \omega_y) \in D$. Then, a distance between two configurations is given by

$$d(x, y) := [\|r_x - r_y\|_2^2 + \|v_x - v_y\|_2^2 + \|\omega_x - \omega_y\|_2^2 + d_R(R_x, R_y)^2]^{1/2}.$$

A distance to a set using the above metric is defined as

$$\|x\|_{\mathcal{M}} := \inf_{y \in \mathcal{M}} d(x, y). \quad (8)$$

The following definition of set-stability is used in this paper. A similar definition can be found using the comparison functions in [20], and the ‘‘almost global’’ property can be found in [19].

Definition 1 A set $\mathcal{M} \subset D$ is

- (1) *stable with respect to (1-4), if for $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that if $\|x(0)\|_{\mathcal{M}} < \delta(\epsilon)$, then $\|x(t)\|_{\mathcal{M}} < \epsilon$,*
- (2) *attractive with respect to (1-4), if there exists an open neighborhood U of \mathcal{M} such that if $x(0) \in U$, then $\lim_{t \rightarrow \infty} \|x(t)\|_{\mathcal{M}} = 0$.*

The set is asymptotically stable if the set is both stable and attractive. In addition, the set is almost globally asymptotically stable if it is asymptotically stable and the set of initial conditions such that $\lim_{t \rightarrow \infty} \|x(t)\|_{\mathcal{M}} \neq 0$ has a zero Lebesgue measure.

4 A New Coordinate System via Near-Identity Diffeomorphism

In this section, a new coordinate for the dynamics in (1-4) is proposed by using a near-identity diffeomorphism. Pick a positive $\alpha \in \mathbb{R}^+$ and consider the following mapping, $\Phi_\alpha : D \rightarrow D$, defined everywhere in the domain,

$$\Phi_\alpha(x) := \begin{pmatrix} \eta_1(x) \\ \eta_2(x) \\ R \\ \omega \end{pmatrix} = \begin{pmatrix} r + \gamma R r_w \\ v + \gamma R(\omega \times r_w) \\ R \\ \omega \end{pmatrix} \quad (9)$$

where $\gamma = 1 + \alpha$. It is obvious that Φ_α is invertible, and the Jacobian of Φ has full rank for all x , and so Φ_α is a global diffeomorphism. If $\alpha = -1$, then Φ_α becomes the identity map; therefore, this is a near-identity diffeomorphism. In this paper, $\alpha > 0$ is assumed. The intuition of this mapping is such that the origin of the body frame is shifted up by $(1 + \alpha)R r_w$, which is the direction of the body z axis. This new virtual control point (VCP) refers to the point above the CoP of the vehicle, shown by the green circle in Fig. 1.

Suppose that the vector field of (1-4) is rearranged to a control affine form with $(f(x), g(x))$ with drift $f(x)$, actuation $g(x)$, and control $u = (T, \tau)$. Observe that $\dot{\eta}_1(x) = \eta_2(x)$, and by defining $\eta_1(x)$ as an output of the system, the dynamics of the new coordinate in (9) can be transformed to an almost normal form (the control term appears in the zero dynamics). The Lie derivatives of η_1 along with f and g (see definitions in [16]) are

$$L_f^2 \eta_1(x) = f_1(R, \omega) - b_w(C_1(R)\eta_2(x) + C_2(R)\omega) \quad (10)$$

$$L_g L_f \eta_1(x) = \begin{pmatrix} R e_3; -\gamma R \hat{r}_w I^{-1} \end{pmatrix}, \quad (11)$$

where $;$ indicates a column-wise concatenation, and

$$f_1(R, \omega) = -g e_3 + \gamma R \hat{\omega}^2 r_w + \gamma R \hat{r}_w I^{-1} (\hat{\omega} I \omega)$$

$$C_1(R) = \left(\frac{1}{m} I_{3 \times 3} - \gamma R \hat{r}_w I^{-1} \hat{r}_w R^\top \right)$$

$$C_2(R) = \alpha R \hat{r}_w \left(\frac{1}{m} I_{3 \times 3} - \gamma I^{-1} \hat{r}_w^2 \right).$$

Let $I_{3 \times 3}$ and $0_{3 \times 3}$ be the identity and zero matrices, and

$$A_c = \begin{pmatrix} 0_{3 \times 3} & I_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} \end{pmatrix}, B_c = \begin{pmatrix} 0_{3 \times 3} \\ I_{3 \times 3} \end{pmatrix}, E = \begin{pmatrix} \bar{0} \\ I_{3 \times 3} \end{pmatrix}.$$

The topologically conjugate system to (1-4) is derived,

$$\dot{\eta} = A_c \eta + B_c (L_f^2 \eta_1 + A(R)u) \quad (12)$$

$$\dot{R} = R \hat{\omega} \quad (13)$$

$$I \dot{\omega} = -\hat{\omega} I \omega - b_w \hat{r}_w R^\top (\eta_2 - \alpha R (\hat{\omega} r_w)) + E u, \quad (14)$$

where $\eta = (\eta_1, \eta_2)^\top$ and $A(R) := L_g L_f \eta_1(x)$. Observe that $L_g L_f \eta_1(x) \in \mathbb{R}^{3 \times 4}$, while not square, is full row rank, since $\text{rank}(\hat{e}_3) = 2$, and $e_3^\top \hat{e}_3 = \bar{0}$. This differs from a typical partial output feedback linearization since the system is not square; however, the row space of $L_g L_f \eta_1(x)$ is orthogonal to $e_4 := (0, 0, 0, 1)^\top$, which indicates that $A(R)u$ in (12) is independent of the yaw torque τ_3 . Therefore, the output dynamics can be fully controllable without a yaw torque input and the output dynamics can be feedback linearizable with three inputs, (T, τ_1, τ_2) . In addition, the ω_3 state in $\Phi(x)$ is fully controllable with τ_3 and is independent of the output dynamics.

5 Stability Analysis of Partial Feedback Linearization Controller

First, a partial feedback linearization controller is designed to exponentially stabilize the output to become zero, $\eta = \bar{0}$. Since $A(R)$ is full row rank, there exists a pseudo inverse, such that $A^\dagger(R) := A(R)^\top (A(R)A(R)^\top)^{-1}$.

5.1 Partial Feedback Linearization Controller

We choose $Q_c \in \mathbb{R}^{6 \times 6}$ and $R_c \in \mathbb{R}^{3 \times 3}$ to be positive definite matrices and pick $k_w > 0$. Since (A_c, B_c) is controllable, there exists a unique positive definite $P \in \mathbb{R}^{6 \times 6}$ that solves the continuous-time algebraic Riccati equation (CARE) using (A_c, B_c, Q_c, R_c) . The linear quadratic regulator (LQR) is then used to stabilize the output dynamics with gain $K := R_c^{-1} B_c^\top P$. Then, the state feedback controller can be defined as

$$u := A^\dagger(R)(-L_f^2 \eta_1(x) - K\eta(x)) - k_w e_4 \omega_3. \quad (15)$$

Note that a simple yaw control law, $\tau_3 = -k_w \omega_3$, in the last term is used but does not appear in the output dynamics in (12) since $A(R)e_4 = \bar{0}$ holds. Now, by substituting u in (12-14) with the controller in (15),

$$\dot{\eta} = (A_c - B_c K)\eta \quad (16)$$

$$\dot{R} = R\hat{\omega} \quad (17)$$

$$\dot{\omega} = h(R, \omega) + g_1(R)\eta, \quad (18)$$

where $h(R, \omega) \in \mathbb{R}^{3 \times 1}$ and $g_1(R) \in \mathbb{R}^{3 \times 6}$ are

$$h(R, \omega) = \begin{pmatrix} \omega_2 \omega_3 \\ -\omega_1 \omega_3 \\ a_1(\omega_1 \omega_2) - k_w \omega_3 \end{pmatrix} + \frac{b_w \alpha}{\gamma} (\hat{e}_3^2 \omega) + \frac{g}{l\gamma} (\hat{e}_3 R^\top e_3) \quad (19)$$

$$g_1(R) = \frac{1}{l\gamma} (\hat{e}_3 R^\top (\frac{b_w}{m} B_c^\top - K)) \quad (20)$$

and $a_1 = (I_{xx} - I_{yy})/I_{zz}$. If $I_{xx} = I_{yy}$, then the zero dynamics (when $\eta_1 = \eta_2 = \bar{0}$) equate to a 3D pendulum in [26] with a rotational damping effect. By using the proposed controller, the aerodynamic drag acting on the CoP is transformed such that it behaves as a rotational damper in the new coordinate system, to eventually help stabilize to the upright orientation.

5.2 Set of Equilibrium Points

There exist two disjoint sets of equilibrium points for the nonlinear system in (16-18),

$$M_{up}^f := \{x \in D | \eta_1 = \eta_2 = \omega = \bar{0}, e_3^\top R^\top e_3 = 1\} \quad (21)$$

$$M_{inv}^f := \{x \in D | \eta_1 = \eta_2 = \omega = \bar{0}, e_3^\top R^\top e_3 = -1\}. \quad (22)$$

Geometrically, the condition for M_{up}^f represents the upright orientation with a free yaw angle, whereas the condition for M_{inv}^f represents the inverted orientation. This is akin to the hanging equilibrium and the inverted equilibrium for a 3D pendulum in [26]. The set of the union of two equilibrium sets is denoted as $M_{eq}^f = M_{up}^f \cup M_{inv}^f$.

5.3 Stability Analysis of the Full Dynamics

In this section, the set-stability of two disjoint equilibrium sets is considered. Let the following function $V : D \rightarrow \mathbb{R}$ be defined as

$$V(x) := \nu^\top P_v \nu + \frac{g}{l\gamma} (1 - e_3^\top R e_3), \quad (23)$$

where $\nu \in \mathbb{R}^9$ and $P_v \in \mathbb{R}^{9 \times 9}$ are defined as

$$\nu = \begin{pmatrix} R(e_3 \times \omega) \\ \eta \end{pmatrix}, \quad P_v = \begin{pmatrix} \frac{1}{2} I_{3 \times 3} & 0_{3 \times 6} \\ 0_{6 \times 3} & k_p P \end{pmatrix}, \quad (24)$$

where $x = (\eta, R, \omega)$, $k_p > 0$, and $0_{3 \times 6}, 0_{6 \times 3}$ are zero matrices. Observe that V is only positive semi-definite since ω_3 does not appear in V . By taking the derivative of V along (16-18),

$$\dot{V}(x) = -\nu^\top P_f \nu, \quad (25)$$

holds for all $\nu \in \mathbb{R}^9$ and $P_f \in \mathbb{R}^{9 \times 9}$ is defined as

$$P_f = \begin{pmatrix} \frac{b_w \alpha}{m\gamma} I_{3 \times 3} & \frac{1}{2l\gamma} (\frac{b_w}{m} B_c^\top - K) \\ \frac{1}{2l\gamma} (\frac{b_w}{m} B_c^\top - K)^\top & k_p Q_c \end{pmatrix}. \quad (26)$$

Observe that P_f is a constant matrix, and it is a function of k_p . Let $S \in \mathbb{R}^{6 \times 6}$ be a positive semi-definite matrix:

$$S := \frac{m}{4b_w l^2 \gamma \alpha} (\frac{b_w}{m} B_c^\top - K)^\top (\frac{b_w}{m} B_c^\top - K). \quad (27)$$

Lemma 2 If $k_p > \lambda_M(S)/\lambda_m(Q_c)$, where $\lambda_M(S)$ is the maximum eigenvalue of S and $\lambda_m(Q_c)$ is the minimum eigenvalue of Q_c , then P_f is positive definite.

PROOF. Since diagonal blocks of P_f are invertible, P_f is positive definite if and only if $k_p Q_c - S$ is positive definite by Theorem 7.76 in [14]. The condition for k_p suffices to show that $k_p Q_c - S$ is positive definite. \square

5.3.1 Boundedness of the trajectories

Now, pick k_p to satisfy Lemma 2, then (25) implies that $\dot{V}(x) \leq 0$ for all $x \in D$. However, the sub-level set of V in (23) is unbounded as V is independent of ω_3 . Therefore, the boundedness of the trajectories needs to be first shown before possibly applying LaSalle's invariance theorem (or Barbalat's lemma) for proving the attractiveness to the desired set. By considering the dynamics of yaw in (18) separately, the following proposition and corollary show that the trajectories of (16-18) are bounded for each initial condition.

Proposition 3 *There exists a class \mathcal{K} function, $\beta_1 : [0, \infty) \rightarrow [0, \infty)$ such that, for any $\epsilon > 0$, if $|\omega_3(0)| < \epsilon$, then $\|\omega(t)\|_2 < \beta_1(\epsilon)$ for all $t \geq 0$.*

PROOF. Suppose that $|\omega_3(0)| < \epsilon$, and pick $c_1 > 0$ such that $V(x(0)) \leq c_1 \epsilon^2$, then $|\omega_1(t)\omega_2(t)| \leq 1/2(\omega_1^2(t) + \omega_2^2(t)) \leq c_1 \epsilon^2$ holds for all $t \geq 0$ since $\dot{V}(x(t)) \leq 0$. By solving for $\omega_3(t)$ in (18), $|\omega_3(t)| \leq \epsilon + a_1 c_1 \epsilon^2 / k_w$ holds. Also, by using the inequality for the l_1 and l_2 norms,

$$\|\omega(t)\|_2 \leq \|\omega(t)\|_1 \leq (2\sqrt{2}c_1 + 1)\epsilon + a_1 c_1 \epsilon^2 / k_w. \quad (28)$$

This is true for all $t > 0$. The right hand side of (28) can be shown to be a class \mathcal{K} function over ϵ , hence the proposition holds. \square

Corollary 4 *For any bounded open neighborhood, U , of M_{eq}^f , there exists a compact subset Ω_c such that if $x(0) \in U$, then $x(t) \in \Omega_c$ for all $t > 0$.*

PROOF. Let $\Omega \in D$ be a closure of U . Since V is continuous, Ω is compact, and $\dot{V}(x(t)) \leq 0$, there exists $V_{max} > 0$ such that $V(\phi(x(t), y)) \leq V_{max}$ for every $y \in \Omega$ where $\phi(x(t), y)$ is a flow starting from y . Now, pick $\epsilon > 0$ such that all $x(0) \in \Omega$ satisfies $|\omega_3(0)| < \epsilon$, then by invoking Proposition 3, $\|\eta(t)\|_2^2 + \|\omega(t)\|_2^2$ is uniformly bounded for all $t \geq 0$ over every $x(0) \in U$. Hence, there exists a compact subset Ω_c such that $x(t) \in \Omega_c$ for all $t > 0$ since $SO(3)$ is compact.

5.3.2 Almost global asymptotic stability of M_{up}^f .

Next, the stability of M_{up}^f is shown as follows.

Theorem 5 *M_{up}^f is a stable set with respect to (16-18).*

PROOF. Let $c_2 = \max(\lambda_M(P_v), g/(l\gamma))$ and $\bar{\epsilon} < \min(1, \lambda_m(P)/c_2)/\sqrt{10}$. Pick $\epsilon > 0$ such that $\epsilon < \bar{\epsilon}$. Suppose that the initial condition satisfies $\|x(0)\|_{M_{up}^f}^2 < \epsilon$, then $V(x(0)) \leq c_2 \epsilon$ holds since

$$1 - e_3^\top R^\top e_3 \leq \inf_{\psi \in [-\pi, \pi]} \text{tr}(I_3 - R^\top R_\psi)$$

holds for all $R \in SO(3)$ and all $e_3^\top R_\psi e_3 = \pm 1$. Since $\dot{V}(x) \leq 0$ for all $x \in D$, $\|\eta(t)\|_2^2 + \|\omega(t)\|_2^2 \leq c_2 \epsilon / \lambda_m(P_v) + \beta_1(\sqrt{\epsilon})^2$ holds for all $t \geq 0$, where β_1 is a class \mathcal{K} function proposed in Proposition 3.

The last step is to make sure the distance from $R(t)$ to the orientations of M_{up}^f (any $R_e \in SO(3)$ satisfying $e_3^\top R_e^\top e_3 = 1$) is bounded by some class \mathcal{K} function over ϵ for all $t \geq 0$. Since R_e is a special orthogonal matrix, R_e can be represented by one parameter set $\psi \in [-\pi, \pi]$, where $e_1^\top R_\psi = [\cos \psi, -\sin \psi, 0]$, $e_2^\top R_\psi = [\sin \psi, \cos \psi, 0]$ and $e_3^\top R_\psi = e_3^\top$. Let (a_1, a_2, a_3) and (b_1, b_2, b_3) denote the first and second columns of R^\top , respectively. Observe that the Frobenious norm of the matrix can be represented by the trace operator, [15], and that

$$\inf_{\psi \in [-\pi, \pi]} d_R(R, R_\psi) = 3 - r - e_3^\top R^\top e_3$$

where $r = \sqrt{(a_1 + b_2)^2 + (a_2 - b_1)^2}$. Since R is in special orthogonal group, $\det(R) = 1$, and $1 - e_3^\top R^\top e_3 < c_2 \epsilon / \lambda_m(P_v)$, r is lower bounded by $r > \sqrt{4 - 40(c_2 \epsilon / \lambda_m(P_v))^2}$. The lower-bound is well defined since $\epsilon < \bar{\epsilon}$ by the assumption. By substituting the lower-bound for r , $\inf_{\psi \in [-\pi, \pi]} d_R(R, R_\psi) \leq \beta_4(\epsilon)$ holds where $\beta_4(\epsilon)$ is a class \mathcal{K} function defined as $2 - \sqrt{4 - 40(c_2 \epsilon / \lambda_m(P_v))^2} + \epsilon$. Hence, $\|x(t)\|_{M_{up}^f}^2 \leq c_2 \lambda_m(P_v) \epsilon + \beta_1(\sqrt{\epsilon})^2 + \beta_4(\epsilon)$ for all $t \geq 0$, and so M_{up}^f is a stable set. \square

Attractiveness to the set of equilibrium, M_{eq}^f , the union of M_{up}^f and M_{inv}^f , is shown in the following proposition.

Proposition 6 *M_{eq}^f is an attractive set with respect to (16-18).*

PROOF. Let U be a bounded open neighborhood of M_{eq}^f such that $x(0) \in U$, and $g : [0, \infty) \rightarrow [0, \infty)$ be a twice differentiable function defined by $g(t) := V(x(t))$

using V in (23). Since $g(t)$ is lower bounded and $\dot{g}(t) \leq 0$ for all $t \geq 0$ by (25), $\lim_{t \rightarrow \infty} \int_0^t \dot{g}(t) dt$ exists and is finite. The second derivative of g , is expressed as $\ddot{g} = -2\nu P_f \dot{\nu}$, where $\nu := (\nu_1, \nu_2)$ such that $\nu_1 := R(e_3 \times \omega)$ and $\nu_2 := \eta$ as given in (24). Observe that $\dot{\nu}_1 = R\hat{\omega}(e_3 \times \omega) + R(e_3 \times \dot{\omega})$, and $\dot{\nu}_2 = \dot{\eta}$, where $\dot{\eta}$ and $\dot{\omega}$ are given in (16-18). Since the trajectory is bounded in some compact set, Ω_c , as shown in Corollary 4, $\ddot{g}(t)$ is also bounded for all $t \geq 0$. Therefore, \dot{g} is uniformly continuous, and by invoking Barbalat's lemma [16], $\lim_{t \rightarrow \infty} \dot{g} = 0$ holds. Since P_f in (26) is chosen to be positive definite by satisfying Lemma 2, $\lim_{t \rightarrow \infty} \|\nu(t)\| = 0$ holds, which infers that $\lim_{t \rightarrow \infty} \|\eta(t)\|_2 = 0$ and $\lim_{t \rightarrow \infty} (\omega_1^2 + \omega_2^2) = 0$. In addition, $\lim_{t \rightarrow \infty} \|v\|_2 = 0$ also holds as $v = \eta_2 + l\gamma\nu_1$ is defined in (9). Let $g_i^v : [0, \infty) \rightarrow \mathbb{R}$ be defined as $g_i^v(t) := v_i(t)$ where v_i is i -th component of v , then $\lim_{t \rightarrow \infty} \int_0^t \dot{g}_i^v(t) dt$ exists and is finite where $\dot{g}_i = e_i^\top \dot{v}$. Also, $\ddot{g}_i(t)$ is bounded for all $t \geq 0$ since $\ddot{g}_i^v := e_i^\top \ddot{v}$, where $\ddot{v}(t) = \ddot{\eta}_2 + l\gamma\ddot{\nu}_1$, is a function of bounded quantities, according to Corollary 4. By invoking Barbalat's lemma, $\lim_{t \rightarrow \infty} \dot{g}_i^v(t) = 0$ holds, and so $\lim_{t \rightarrow \infty} \|\dot{v}(t)\|_2 = 0$ holds. Observe that

$$\lim_{t \rightarrow \infty} (-\gamma R \hat{e}_3^2 R^\top e_3) = \lim_{t \rightarrow \infty} \dot{v}(t) = 0$$

holds since $\lim_{t \rightarrow \infty} \|\eta\|_2 = 0$ and $\lim_{t \rightarrow \infty} (\omega_1^2 + \omega_2^2) = 0$. Therefore, $\lim_{t \rightarrow \infty} e_3 \times R^\top e_3 = 0$ holds, or equivalently, $\lim_{t \rightarrow \infty} e_3^\top R e_3 = 1$ or -1 . Now, pick $\epsilon < \bar{\epsilon}$ as in the proof of Theorem 5, then there exist $T_1 > 0$ such that if $t > T_1$, then $|\omega_1(t)\omega_2(t)| \leq \epsilon$ and $\|x(t)\|_{M_{eq}^f}^2 \leq \epsilon + \omega_3(t)^2$ holds. By invoking the comparison lemma (Lemma 3.4 in [16]) to the dynamics for ω_3 in (18), there exist $c_3 > 0$ and $T_2 > T_1$ such that if $t > T_2$, then $|\omega_3(t)| < c_3\epsilon$, and by invoking Proposition 3, $\|x(t)\|_{M_{eq}^f}^2 < \epsilon + c_3^2\beta_1(\epsilon)^2$ for all $t > T_2$. Hence, $\lim_{t \rightarrow \infty} \|x(t)\|_{M_{eq}^f}^2 = 0$ for all $x(0) \in U$. \square

The first main theorem follows.

Theorem 7 M_{up}^f is an almost globally asymptotically stable set with respect to (16-18).

PROOF. It is shown that M_{up}^f is locally stable in Theorem 5 and M_{eq}^f is globally attractive in Proposition 6. First the linearization around M_{inv}^f is considered. Pick $(\bar{0}, \bar{0}, R_{inv}, \bar{0}) \in M_{inv}^f$, and linearize (16-18) around this point. Observe that the linearized model for (R, ω) can be derived into a second order differential equation $(x_1, x_2, x_3) \in \mathbb{R}^3$,

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} -\frac{b_w \alpha}{\gamma} \dot{x}_1 + \frac{g}{l\gamma} x_1 \\ -\frac{b_w \alpha}{\gamma} \dot{x}_2 + \frac{g}{l\gamma} x_2 \\ 0 \end{pmatrix}. \quad (29)$$

This is similar to the linearization around an inverted equilibrium for a 3D pendulum in [3] except that (29) has a damping term. There are two positive eigenvalues, two negative eigenvalues, and two zero eigenvalues. Since there are two positive eigenvalues, M_{inv}^f is an unstable set. Now, it is enough to show that the region of attraction for M_{inv}^f is a set of measure zero. The two zero eigenvalues correspond to the yaw rotation which is invariant in M_{inv}^f , and so the center manifold is contained in M_{inv}^f . Therefore, all the trajectories near $(\bar{0}, \bar{0}, R_{inv}, \bar{0})$ will diverge except the two-dimensional stable submanifold. This holds for any $(\bar{0}, \bar{0}, R_e, \bar{0}) \in M_{inv}^f$, so the stable submanifold has Lebesgue measure zero [17]. A similar argument can be found for the smooth control design in [4]. Hence, M_{up}^f is an AGAS set. \square

6 Adaptive Controller for a Bounded Uncertain Drag Coefficient

The AGAS property of the partial feedback linearizing controller was proven assuming perfect knowledge of the drag coefficient, which is not practically desirable. Nevertheless, it is interesting to observe that for any given $b_w > 0$, there exists at least one partial feedback linearizing controller to make the set, M_{up}^f , AGAS. By exploiting this fact, an adaptive controller is designed for a bounded uncertain drag coefficient. Assume that $b_w \in I_b$, where $I_b := [\underline{b}_w, \bar{b}_w]$, is bounded with $\underline{b}_w, \bar{b}_w > 0$. Now consider the following controller

$$u(x) := A^\dagger(R)(-L_f^2 \eta_1(x, \hat{b}_w) - K\eta(x)) - k_w e_4 \omega_3 \quad (30)$$

$$\dot{\hat{b}}_w := \begin{cases} 0 & \text{if } f_4(x) < 0 \text{ and } \hat{b} = \underline{b}_w \\ 0 & \text{if } f_4(x) > 0 \text{ and } \hat{b} = \bar{b}_w \\ \Gamma f_4(x) & \text{otherwise} \end{cases} \quad (31)$$

where $\Gamma > 0$ is the adaptation gain, and $L_f^2 \eta_1(x, \hat{b}_w)$ is the same as in (10), with \hat{b}_w instead of b_w ,

$$f_4(x) := -2k_p^* \eta^\top P B_c (C_1(R) \eta_2 / m + C_2(R) \omega) - \omega^\top I^{-1} \hat{r}_w R^\top (\eta_2 - \alpha R(\hat{\omega} r_w)),$$

and with initial condition, $\hat{b}_w(0) \in I_b$.

Theorem 8 M_{up}^f is an almost globally asymptotically stable set with respect to the closed loop dynamics with the adaptive controller in (30-31).

PROOF. Since the S matrix in (27) is a smooth function over b_w , there exists $\lambda_M^*(S) := \max_{b_w \in I_b} \lambda_M(S(b_w))$. Now, pick $k_p^* > \lambda_M^*(S)/\lambda_m(Q)$; then, by Lemma 2, $P_f(\hat{b}_w)$ from (26) is a positive definite matrix for any $\hat{b}_w \in I_b$. Now, consider $V_1 : D \times I_b \rightarrow \mathbb{R}$ defined as

$V_1(x, \hat{b}_w) := V(x) + \Gamma^{-1} \tilde{b}_w^2/2$ where $\tilde{b}_w = b_w - \hat{b}_w$ and $V(x)$ is (23). By taking the derivative,

$$\dot{V}_1 = -\nu^T P_f(\hat{b}_w) \nu + f_4(x) \tilde{b}_w + \Gamma^{-1} \tilde{b}_w \dot{\tilde{b}}_w \quad (32)$$

holds, and by substituting the adaptation law in (31), $\dot{V}_1(x, \hat{b}_w) \leq -\nu^T P_f(\hat{b}_w) \nu$ holds for all $x \in D$ as $f_4(x) \tilde{b}_w \leq 0$ for $\hat{b}_w = \underline{b}$ and $\hat{b}_w = \bar{b}$. Observe that Proposition 3 and Corollary 4 still hold since $V(x) \leq V_1(x, \hat{b}_w)$ and $\dot{V}(x(t)) \leq 0$ for all $t > 0$. Therefore, M_{up}^f is also a stable set, using the same argument as in Theorem 5. Attractiveness to the equilibrium set, M_{eq}^f , is proven differently. Let $\lambda_m^*(P_f)$ be defined as $\min_{\hat{b}_w \in I_b} \lambda_m(P_f(\hat{b}_w))$ where $\lambda_m(P)$ is the minimum eigenvalue of P , then $\|\nu(t)\|_2^2 \leq -\dot{V}_1/\lambda_m^*(P_f)$ holds for all $t \geq 0$. Therefore, ν is in the \mathcal{L}^2 space since

$$\int_0^\infty \nu(t)^\top \nu(t) dt \leq (V_1(0) - V_1(\infty))/\lambda_m^*(P_f)$$

where $V_1(\infty) := \lim_{t \rightarrow \infty} V_1(x(t), \hat{b}_w(t))$, which is well defined as V_1 is lower bounded. In addition, $\|\nu(t)\|_2^2 < V_1(x(0), \hat{b}_w(0))/\lambda_m^*(P_f)$ for all $t \geq 0$ indicates that ν is in the \mathcal{L}^∞ space as well. Since \tilde{b}_w is bounded by the projection-based adaptive rule in (31), $\dot{\nu}(t)$ can be shown to be a bounded function as well. Therefore, $\lim_{t \rightarrow \infty} \|\nu\|_2 = 0$ by invoking the corollary of Barbalat's Lemma (Corollary 2.9 in [23]). Now the same argument of acquiring $\lim_{t \rightarrow \infty} |\dot{\nu}(t)|_2 = 0$ holds as \tilde{b}_w is bounded, which infers that $\lim_{t \rightarrow \infty} e_3^T R e_3 = 1$ or -1 . Finally, considering the ultimate bound for $\omega_3(t)$ as in the proof of Proposition 6, M_{eq}^f can be shown to be asymptotically attractive.

Finally, the linearization around M_{inv}^f on each $\hat{b}_w \in I_b$ exhibits two negative eigenvalues, indicating instability of M_{inv}^f with the Lebesgue measure zero stable submanifold, as in Theorem 7. Hence, M_{up}^f is an AGAS set. \square

7 Application to VCP Reference Trajectory Tracking

In this section, a direct application to the adaptive AGAS controller in (30) is considered to make the VCP follow a reference trajectory. Suppose that a continuously differentiable trajectory in the \mathbb{R}^3 space is given as $\phi \in C^2(\mathbb{R}, \mathbb{R}^3)$, where $C^2(\mathbb{R}, \mathbb{R}^3)$ is a set of continuously twice differentiable functions. Assume that the derivatives of the trajectory are bounded by some $M > 0$, such that $\|\dot{\phi}(t)\|_2 \leq M$ and $\|\ddot{\phi}(t)\|_2 \leq M$.

The objective is to make the VCP of the vehicle follow the reference trajectory. Defining error dynamics as $\xi_1 = \eta_1 - \phi$ and $\xi_2 = \eta_2 - \dot{\phi}$, and with $\xi = (\xi_1, \xi_2) \in \mathbb{R}^6$,

then a feedback controller using ξ is proposed:

$$u := A^\dagger(R)(-L_f^2 \eta_1(x, \hat{b}_w) - K\xi(x) + \ddot{\phi}) - k_w e_4 \omega_3, \quad (33)$$

where $L_f^2 \eta_1(x, \hat{b}_w)$ is defined as in the controller in (30) and the adaptation rule in (31) is modified such that only η in $f_4(x)$ is replaced with ξ and not η_2 . For any given set point, $\phi_0 \in \mathbb{R}^3$, by setting $\phi(t) = \phi_0$, the error $\xi(t)$ almost globally exponentially converges to $\bar{0}$, and $\omega(t)$ almost globally asymptotically converges to $\bar{0}$ using the above controller.

7.1 VCP Trajectory following

The VCP error dynamics for ξ using (33) are

$$\begin{aligned} \dot{\xi} &= (A_c - B_c K)\xi - \tilde{b}_w B_c \epsilon(\xi, R, \omega) \\ \epsilon(\xi, R, \omega) &= (C_1(R)(\xi_2 + \dot{\phi}) + C_2(R)\omega). \end{aligned} \quad (34)$$

Suppose that the gain K is chosen by solving CARE of $(A_c + k_c I_{6 \times 6}, B_c, Q_c, R_c)$ for $k_c > 0$, then all the real parts of the eigenvalues of $(A_c - B_c K)$ are upper bounded by $-k_c$. Let V_1 be defined as in Theorem 8 by replacing η with ξ , then the derivative is computed as

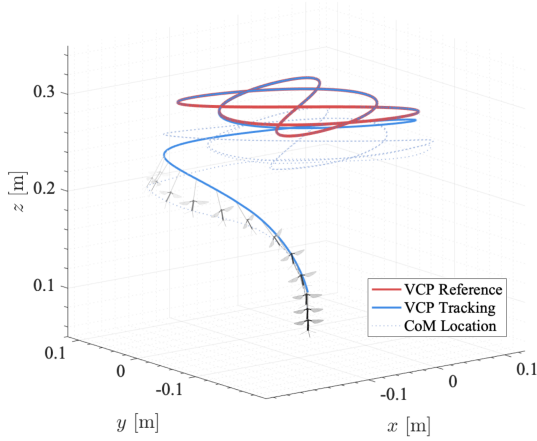
$$\dot{V}_1(x) \leq -c\|\nu\|_2^2 - k_c \lambda_m(P) \|\xi\|_2^2 - \frac{1}{ml\gamma} \langle \nu_1, \hat{b}_w \dot{\phi} + m\ddot{\phi} \rangle,$$

where $c = \lambda_m^*(P_f)$, and $x \in (\xi, R, \omega)$, and $\nu = (\nu_1, \nu_2) \in \mathbb{R}^9$ is redefined with $\nu_1 = R(e_3 \times \omega)$ and $\nu_2 = \xi$. In addition, the upper bounds for \dot{V} can be formulated as

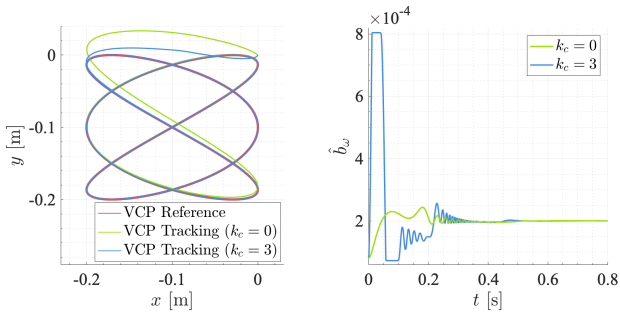
$$\begin{aligned} \dot{V}_1(x) &\leq -(c + k_c \lambda_m(P)) \|\xi\|_2^2 - \|\nu_1\|_2 (c \|\nu_1\|_2 - \mu(M)) \\ &\leq -(c + k_c \lambda_m(P)) \|\xi\|_2^2 \quad \text{if } \|\nu\|_2^2 \geq \frac{2\mu(M)^2}{c^2}, \end{aligned}$$

where $\mu(M) = (\bar{b}_w + m)/(ml\gamma)M$. Applying the above inequalities, it can be shown that there exists $T > 0$ such that if $t > T$, then $\|\nu\|_2 < 2g/(l\gamma) + 2\lambda_M(P_v)\mu(M)^2/(c^2\lambda_m(P_v))$. Therefore, $\|\xi\|_2$ and $\|\omega\|_2$ are ultimately bounded.

The extra control gain, k_c , in addition to the LQR parameters, R_c and Q_c , is used to directly lower the real part of the eigenvalue of $(A_c - B_c K)$ in (34). In the ideal case (when \tilde{b}_w is zero or equivalently using the partial feedback linearization controller), the tracking error, $\|\xi\|_2$, converges exponentially to zero with a guaranteed decaying rate of $-k_c$. A case study with two different k_c gains shows that the parameter error becomes relatively small (within 0.02% of its true value, for both cases), and a faster convergence to the same error bound is achieved by higher k_c . The detailed analysis on the relation between parameter convergence and the tracking performance remains the subject of future work.



(a) $t < 10.0(s)$.



(b) VCP trajectory in (x, y) . (c) Parameter estimation.

Fig. 2. Trajectories using different gains $k_c = 0, 5$.

7.2 Simulation Results

In this section, a tracking controller is applied to follow a Lissajous curve. The desired trajectory, $\phi(t) \in \mathbb{R}^3$, is

$$\phi_x(t) := r_d \cos(a\omega_f t + \delta) \quad (35)$$

$$\phi_y(t) := r_d \cos(b\omega_f t) \quad (36)$$

$$\phi_z(t) := 0.3, \quad (37)$$

where $r_d = 0.1$, $a = 2$, $b = 3$, and $\omega_f = 2\pi/T_{period}$, with $T_{period} = 1$ s. The model parameters for the Harvard RoboBee are given in [11], including $b_w = 2.0 \times 10^{-4}$ Ns m⁻¹. The bounds for the drag coefficients are given as $I_b = [0.4b_w, 4b_w]$ allowing higher upper bound. The tracking controller in (33) is applied, with the gain K designed using $Q_c = \text{diag}(100, 100, 100, 10, 10, 10)$ and $R_c = 0.01I_{3 \times 3}$, with two different gains, $k_c = 0$ and $k_c = 3$. The adaptive gain is chosen with $\Gamma = 0.3$ for (31). A small feedback gain $k_w = I_{zz} = 0.45 \times 10^{-9}$ is chosen for yaw torque control. In this example, $\alpha = 3$ is selected (such that the VCP of the vehicle is four times higher than the CoP position). The chosen initial states are $r = (0, 0, 0.05)$, $v = \omega = \bar{0}$, and $R(0)$ is represented by the axis-angle representation, with angle $\theta_0 = \pi/4$ and axis $(1, 0, 0)$.

The simulation results are shown in Fig. 2a. The tra-

jectory of the VCP using the gain $k_c = 3$ is shown in blue, with the reference trajectory (a Lissajous curve at $z = 0.3$) shown in red. The CoM of the vehicle during VCP tracking is shown with a dotted blue line. Fig. 2a shows a forward simulation over 10.0 seconds (with overlaid orientations for the RoboBee shown over the first 0.5 seconds), demonstrating attitude changes as the VCP approaches the reference trajectory. A projection to the (x, y) plane is shown in Fig. 2b, where the red curve represents the reference curve, the green curve corresponds to the VCP trajectory for $k_c = 0$, and the blue curve corresponds to $k_c = 3$. The estimated drag coefficients are shown in Fig. 2c for both k_c gains. The parameter error is bounded to 0.02% of its true value after 0.5 sec.

8 Conclusions and Discussion

In this paper, a new coordinate system for a flapping-wing vehicle is proposed, which takes advantage of the averaged aerodynamic drag acting on the vehicle. A state feedback controller is designed to regulate a virtual control point (VCP) using a near-identity diffeomorphism which permits partial feedback linearization. The stable zero dynamics resembles the 3D pendulum dynamics where VCP serve as a pivot. The almost global asymptotic stability of the upright configurations is shown for the closed loop system. Next, an adaptive controller is proposed, to preserve the AGAS property given the bounded uncertainty of the drag coefficient. Finally, a VCP tracking controller is designed to achieve ultimate boundedness.

One of the advantages of the VCP based control framework is that the controller requires relatively small yaw torque as the VCP output dynamics is independent of the yaw torque control. This is beneficial to FWMAV control as yaw torque is often a particularly weak torque axis for [2,9,31,13]. In addition, the proposed adaptation algorithm in (31) can be naturally combined with a deadzone method [23] to enhance the robustness to the bounded unmodeled disturbance.

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